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Abstract— **The paper provides computation-oriented necessary and sufficient conditions for the global exponential stability of linear systems with asynchronous sensors and actuators. Precisely, we focus on continuous-time linear systems whose state undergoes finite jumps referred to as impulsions. The impulsions are of two types: those related to input updates and those related to measurement updates. We assume that impulsions of each type occur periodically but the periods may be different and the clocks at the sensor and at the actuator are not synchronized. We first show that the analysis can be reduced to a finite time domain. Based on that, we provide necessary and sufficient conditions for the global exponential stability of systems belonging to the class under study. An example illustrates numerically the proposed results.**

I. INTRODUCTION

The analysis of impulsive systems is motivated by sampled data systems [4], [10], electronic circuits with diodes [18], [17] or mechanical systems with impacts and dry friction [5], [7], [15]. Stability and performances of impulsive systems received a lot of attention as illustrated by the following recent works [2], [16], [1] and the references therein. Stability and stabilization of linear impulsive systems with quasi-periodic impulsions have been considered in [11]. Furthermore, we proposed in [6] a constructive methodology to get necessary and sufficient conditions for the stability of this class of systems.

As pointed out in [19], in real applications we can encounter situations where the local clocks at the sensor and at the actuator are not synchronized. As mentioned there, clocks synchronization over networks has fundamental limitations [8] and GPS synchronization may be vulnerable against malicious attacks [12]. In this context [19] addresses the problem of how large the clock offset between actuator and sensor can be, without compromising the existence of a stabilizing linear time-invariant controller.

In this work we consider a dynamic linear controller that regulates a linear plant. The sensor measurements and the control inputs can be updated periodically but with different sampling period. In this framework, we give computation oriented conditions which are necessary and sufficient for stability of the closed loop. The results provide also a preliminary analysis to the more realistic case in which the sampling periods are not only different, but uncertain within bounded known intervals. The conditions presented in this paper are then necessary for robust stability and the analysis method is extendable, in our opinion, to the uncertain case.

The problem is formulated in terms of global stability analysis of a subclass of hybrid systems studied in [9]. The main contribution of this work is that for this specific subclass we are able to highlight necessary and sufficient conditions characterizing the global uniform exponential stability. Furthermore, our main results provide a numerical methodology to check these necessary and sufficient conditions.

The paper is organized as follows. Section II presents the problem statement and the framework assumptions. Preliminary results allowing to reduce stability analysis to a finite time-domain are provided in Section III. The main results on the constructive necessary and sufficient conditions for the global exponential stability of the system at hand, are developed in Section IV. We illustrate the methodology with one numerical example in Section V before giving some concluding remarks.

This work was partially funded by the ANR project COMPACS - "Computation Aware Control Systems", ANR-13-BS03-004.

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II. PROBLEM FORMULATION

In this paper, we consider a continuous-time linear plant

$$\begin{cases} \dot{x}_p(t) = A_p x_p(t) + B_p u(t), \\ y(t) = C_p x_p(t), \end{cases} \tag{1}$$

where $x_p(t) \in \mathbb{R}^{n_p}$ is the state, $u(t) \in \mathbb{R}^m$ the control input and $y(t) \in \mathbb{R}^{n_y}$ the plant output. The plant is in closed loop with a dynamic controller of the form

$$\begin{cases} \dot{x}_k(t) = A_k x_k(t) + B_k y(t), \\ u(t) = C_k x_k(t), \end{cases} \tag{2}$$

where $x_k(t) \in \mathbb{R}^{n_k}$ is the controller's state. We investigate the scenario where the actuator and the sensor are implemented on a digital platform with zero-order-hold devices. Moreover the updates of the control signal and the measured output are done periodically but independently, i.e. we don't assume clocks synchronization and same period for the input and output of the plant. In order to formalize that, we consider $T_1, T_2 \in \mathbb{R}_+$ the sampling periods of u and y, respectively and we define the following sequences

$$\mathscr{T}_1 = \Big\{ \{t_i\}_{i\in\mathbb{N}} : \ t_{i+1} = t_i + T_1, \ t_0 \in [0, T_1], \ \forall i \in \mathbb{N} \Big\},$$
$$\mathscr{T}_2 = \Big\{ \{t_j\}_{j\in\mathbb{N}} : \ t_{j+1} = t_j + T_2, \ t_0 \in [0, T_2], \ \forall j \in \mathbb{N} \Big\}.$$

Therefore, closed-loop dynamics (1)-(2) becomes

$$\begin{cases} \dot{x}_p(t) = A_p x_p(t) + B_p v(t), \\ y(t) = C_p x_p(t), \\ \dot{x}_k(t) = A_k x_k(t) + B_k w(t), \\ u(t) = C_k x_k(t), \end{cases} \tag{3}$$

where v and w are defined as

$$v(t) = u(t_j), \quad \forall t \in [t_j, t_{j+1}), \ \forall t_j \in \mathscr{T}_1,$$
$$w(t) = y(t_h), \quad \forall t \in [t_h, t_{h+1}), \ \forall t_h \in \mathscr{T}_2.$$

In other words, considering the state vector $x(t) = (x_p(t), w(t), x_k(t), v(t))$, the closed-loop dynamics can be formulated as a continuous-time linear system affected by impulses at time instants $t_j \in \mathscr{T}_1 \cup \mathscr{T}_2$. The continuous-time dynamics is defined by:

$$\begin{cases} \dot{x}_p(t) = A_p x_p(t) + B_p v(t), \\ \dot{w}(t) = 0, \\ \dot{x}_k(t) = A_k x_k(t) + B_k w(t), \\ \dot{v}(t) = 0, \end{cases} \quad \forall t \in \mathbb{R}_+ \setminus (\mathscr{T}_1 \cup \mathscr{T}_2) \tag{4}$$

while the impulses are given by one of the following equations:

$$\begin{cases} x_p(t_j) = x_p(t_j^-), \\ w(t_j) = w(t_j^-), \\ x_k(t_j) = x_k(t_j^-), \\ v(t_j) = C_k x_k(t_j^-), \end{cases} \quad \forall j \in \mathscr{T}_1 \tag{5}$$

and

$$\begin{cases} x_p(t_j) = x_p(t_j^-), \\ w(t_j) = C_p x_p(t_j^-), \\ x_k(t_j) = x_k(t_j^-), \\ v(t_j) = v(t_j^-), \end{cases} \quad \forall j \in \mathscr{T}_2 \tag{6}$$

where, for any function $f(t)$ we introduce the notation $f(t^-) = \lim_{\tau \to 0, \tau < 0} f(t + \tau)$. Summarizing, the overall asynchronously sampled system can be modelled as the following linear impulsive system:

$$\begin{cases} \dot{x}(t) = A_c x(t), & \forall t \in \mathbb{R}_+ \setminus (\mathscr{T}_1 \cup \mathscr{T}_2), \\ x(t) = A_1 x(t^-), & \forall t \in \mathscr{T}_1, \\ x(t) = A_2 x(t^-), & \forall t \in \mathscr{T}_2, \end{cases} \tag{7}$$

where $x \in \mathbb{R}^n$ is the system state, i.e. $x = (x_p, w, x_k, v)$ with $n = n_p + m + n_k + p$, and A_c, A_1 and A_2 are appropriate matrices defined from (4)-(6). Dynamics (7) can be also obtained in the framework of multi-agent systems that update asynchronously their inputs. We will refer to jump (or impulse) of type 1 (resp. 2) if the discrete-time dynamics is defined by the matrix A_1 (resp. A_2).

Remark 1:
- Throughout this paper we suppose that $T_1 \leq T_2$. This would not entail any loss of generality, since a simple redefinition of the system structure could be done if necessary.
- We don't exclude the case where $t \in \mathscr{T}_1 \cap \mathscr{T}_2$. In this case the state-jump is defined by both A_1 and A_2. Furthermore, the jump at $t \in \mathscr{T}_1 \cap \mathscr{T}_2$ is defined either as $x(t) = A_1 A_2 x(t^-)$, or $x(t) = A_2 A_1 x(t^-)$ in order to ensure the outer semi-continuity of the jump map (see [9] for details).

A hybrid model of the systems (7) can be directly given by introducing two additional variables τ_1 and τ_2 taking into account the time passed from the last jump of type 1 or 2, respectively, flow and jump maps and sets. That is, denoting $z = (x, \tau_1, \tau_2) \in \mathbb{R}^{n+2}$ define

$$\begin{cases} \dot{x}(t) = A_c x(t), \\ \dot{\tau}_1(t) = 1, \quad \text{if } z \in C, \\ \dot{\tau}_2(t) = 1, \end{cases} \quad \{ z^+ \in G(z), \quad \text{if } z \in D \tag{8}$$

where $C = \mathbb{R}^n \times [0, T_1] \times [0, T_2]$, $D = \mathbb{R}^n \times \{T_1\} \times \{T_2\}$ and

$$G(z) = \begin{cases} (A_1 x, 0, \tau_2), & \text{if } \tau_1 = T_1, \\ (A_2 x, \tau_1, 0), & \text{if } \tau_2 = T_2. \end{cases} \tag{9}$$

Notice that, while the continuous-time dynamics is given by a function, $G(z)$ is a set-valued map, as $\tau_1 = T_1$ and $\tau_2 = T_2$ implies that both jumps occurs concomitantly yielding

$$z^+ \in \{(A_1 A_2 x, 0, 0), \ (A_2 A_1 x, 0, 0)\} \text{ if } \tau_1 = T_1, \ \tau_2 = T_2.$$

In this work we deal with the case of commensurate sampling periods. Therefore, throughout the paper, we suppose that the following assumption holds true.

Assumption 1: There exist $n_1, n_2 \in \mathbb{N}$ such that $T_1 n_1 = T_2 n_2$ and n_1/n_2 is irreducible.

This first assumption imposes the commensurability of T_1 and T_2 and introduces the minimal pair $n_1, n_2 \in \mathbb{N}$ defining the ratio T_2/T_1. Under this assumption we investigate the global exponential stability of system (7) regardless the initialization of $(\tau_1(0), \tau_2(0)) \in [0, T_1] \times [0, T_2]$. In other words we want to solve the following equivalent problem.

Problem 1: Find necessary and sufficient conditions under which the compact set

$$\mathscr{A} = \{0\} \times [0, T_1] \times [0, T_2], \tag{10}$$

is globally exponentially stable with respect to the hybrid dynamics (8).

The main difference with respect to [9] is that we give constructive necessary and sufficient condition that can be numerically checked.

Remark 2: Considering the case of commensurate sampling periods with different clock initializations $(\tau_1(0), \tau_2(0)) \in [0, T_1] \times [0, T_2]$ is a preliminary analysis for the more general problem in which the sampling periods are not constant but they belong to given bounded intervals. Notice that the general case encloses the one of fixed but not commensurate sampling periods. Moreover, the problem studied here leads to conditions that result to be necessary for the non-constant periods case.

III. Analysis of the time domain

In this section we show that the analysis can be reduced to some finite time domains included in $[0, T_1) \times [0, T_2)$. First, notice that, since we give global asymptotic stability results, the analysis can always be reduced to the case in which the first jump does not occur at $t = 0$. This is equivalent to consider that the following assumption holds.

Assumption 2: Suppose that $\tau_1(0) \in [0, T_1)$ and $\tau_2(0) \in [0, T_2)$.

Throughout the rest of the paper we denote $\bar{\tau} = (\tau_1(0),\, \tau_2(0))$.

A. Geometrical illustration of the jumps sequence

Definition 1: Consider the system (8)-(9) with $\tau_1(0) = \bar{\tau}_1 \in [0,\ T_1)$, $\tau_2(0) = \bar{\tau}_2 \in [0,\ T_2)$. Let T_1, T_2 satisfy Assumption 1 and $D = T_1 n_1 = T_2 n_2$, $n_D = n_1 + n_2$, $T \in [0, D]$. We define

$$\begin{aligned}
&\mathscr{T}_{1,T}(\bar{\tau}) = \{t \in \mathbb{R}_+ :\ (t - \bar{\tau}_1) \bmod T_1 = 0,\\
&\qquad\qquad \wedge\ (t - \bar{\tau}_1) \le T\} \in [0,\ T]\},\\
&\mathscr{T}_{2,T}(\bar{\tau}) = \{t \in \mathbb{R}_+ :\ (t - \bar{\tau}_2) \bmod T_2 = 0,\\
&\qquad\qquad \wedge\ (t - \bar{\tau}_2) \le T\} \in [0,\ T]\},\\
&n_{1,T}(\bar{\tau}) = |\mathscr{T}_{1,T}(\bar{\tau})|, \quad n_{2,T}(\bar{\tau}) = |\mathscr{T}_{2,T}(\bar{\tau})|,\\
&n_T(\bar{\tau}) = n_{1,T}(\bar{\tau}) + n_{2,T}(\bar{\tau}),\\
&\mathscr{T}_T(\bar{\tau}) = \{\{t_k\}_{k \in \mathbb{N}_{n_T}} :\ t_k \le t_{k+1},\ \forall k \in \mathbb{N}_{n_T - 1}\\
&\qquad\qquad \wedge\ \ t_k \in \mathscr{T}_{1,T} \cup \mathscr{T}_{2,T},\ \forall k \in \mathbb{N}_{n_T}\},\\
&\mathscr{T}(\bar{\tau}) = \lim_{T \to +\infty} \mathscr{T}_T(\bar{\tau}).
\end{aligned} \tag{11}$$

Roughly speaking: $\mathscr{T}_{i,T}(\bar{\tau})$ is the sequence of instants of jumps of type i in the interval $[0, T]$ and $n_{i,T}(\bar{\tau})$ its length, with $i \in \mathbb{N}_2$; $\mathscr{T}_T(\bar{\tau})$ the overall sequence of jump instants. Notice that two jump instants can coincide, i.e.t_k could be equal with t_{k+1}.

Let us illustrate the set $\mathscr{T}_D(\bar{\tau})$ (i.e. $\mathscr{T}_T(\bar{\tau})$ in (11) when $T = D$) with the following example.

Example 1: Consider the system (8) with (9) and focus on the dynamics of the variables τ_1 and τ_2, that are not affected by the value of x. Let us suppose for this illustration that $\tau_1(0) = \tau_2(0) = 0$. We consider that Assumption 1 holds with $T_1/T_2 = n_2/n_1 = 7/3$, say $T_1 = 7a$ and $T_2 = 3a$ with $a > 0$. Thus, as can be noticed by graphical inspection in Figure 1, left, the sequence of the first 8 jumps whose time is denoted as t_i with $i \in \mathbb{N}_8$, is

$$\begin{aligned}\{(t_1, 2),\ (t_2, 2),\ (t_3, 1),\ (t_4, 2),\ (t_5, 2),\\ (t_6, 1),\ (t_7, 2),\ (t_8, 2)\}\end{aligned} \tag{12}$$

while at time t_9 both the first and the second jump occur, that means that the subsequence (12) might be followed either by $\{(t_9, 1),\ (t_{10}, 2)\}$ or $\{(t_9, 2),\ (t_{10}, 1)\}$ with $t_9 = t_{10}$. The values of t_i are

$$\{t_i\}_{i \in \mathbb{N}_{10}} = \{3a, 6a, 7a, 9a, 12a, 14a, 15a, 18a, 21a, 21a\} \tag{13}$$

and $\tau_1(21a) = \tau_2(21a) = 0 = \tau_1(0) = \tau_2(0)$. This yields that, at the time instant which is the minimal common multiplier of n_1 and n_2, the partial state (τ_1, τ_2) recovers its initial value. Notice that $t_{h_1+1} = t_{h_1} + 7a$ and $t_{h_2+1} = t_{h_2} + 3a$ as required,

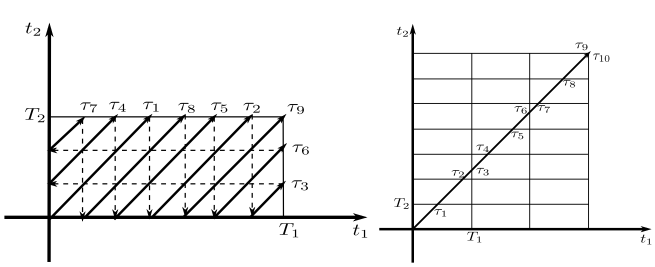

Fig. 1. Graphical construction of the jump sequence.

for all $h_1 \in \mathscr{T}_{1,D}$ and $h_2 \in \mathscr{T}_{2,D}$. Thus, the jump sequence is given by the concatenation of the subsequence (12) and the jump instants are

$$\begin{aligned}\mathscr{T} = \{t_i\}_{i \in \mathbb{N}} = \Big\{&\{3a + 21ak,\, 6a + 21ak,\, 7a + 21ak,\\ &9a + 21ak,\, 12a + 21ak,\, 14a + 21ak,\, 15a + 21ak,\\ &18a + 21ak,\, 21a + 21ak,\, 21a + 21ak\}\Big\}_{k \in \mathbb{N}}.\end{aligned}$$

An analogous graphical illustration of the sequence of jumps is illustrated in Figure 1, right. Since Assumption 1 is satisfied, the set in the space of $(\tau_1,\, \tau_2) \in \mathbb{R}^2$ can be constructed by replicating the basic set $[0, 7a] \times [0, 3a]$ to obtain a square in $\mathbb{R}^2$, in particular $[0, 21a] \times [0, 21a]$. Notice that the jump instants are given by the points in $(\tau_1,\, \tau_2) \in \mathbb{R}^2$ such that $\tau_1 = 7ak$ and $\tau_2 = 3ak$ for all $k \in \mathbb{N}$. Moreover, analogous considerations hold for the case of $(\tau_1(0),\ \tau_2(0)) \neq (0,\ 0)$.

B. Finite time-domain reduction

The sets and sequences in Definition 1 permit to determine a cyclic behavior of the hybrid system (8) with initial time variable $\bar{\tau}$.

Proposition 1: Given the system (8) and (9) satisfying Assumption 1, the sequence of jumps is

$$\mathscr{T}(\bar{\tau}) = \{\mathscr{T}_D(\bar{\tau}),\ \mathscr{T}_D(\bar{\tau}),\ \ldots\}. \tag{14}$$

Proof: The result follows directly from the construction of the structures defined in (11). First notice that $\mathscr{T}_{1,D}(\bar{\tau})$ and $\mathscr{T}_{2,D}(\bar{\tau})$ are the time instants in the interval $(0, D]$ at which the first and the second type impulses occur, respectively. Thus, $\mathscr{T}_D(\bar{\tau})$ is the sequence of jump time instants in $[0, D]$. Moreover, from Assumption 1, the value of τ_1, and τ_2, after a time interval of length D are $\bar{\tau}_1$ and $\bar{\tau}_2$, respectively, and n_2 jumps of type 1 and n_1 jumps of type 2 have occurred, see Figure 2, left. Therefore, the subsystem whose state is $(\tau_1,\ \tau_2)$ returns to the initial value after n_2 jumps 1 and n_1 jumps 2 after D, and then the same happens after $2D$, and kD with $k \in \mathbb{N}$ in general. ■

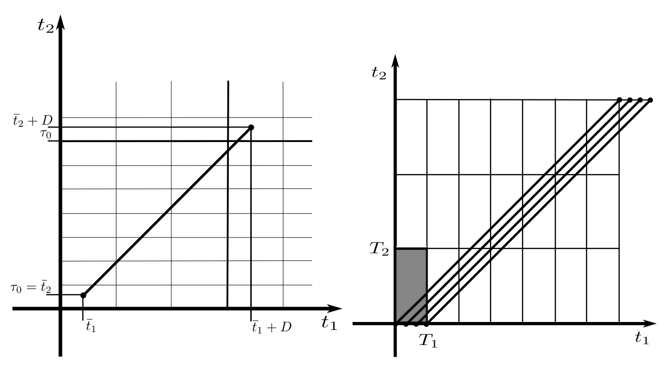

Fig. 2. Graphical construction of the jump sequence

To characterize the dynamics generated in function of the initial condition $\bar{\tau} \in [0, T_1) \times [0, T_2)$ we first discriminate the case in which $\tau_1 = T_1$ and $\tau_2 = T_2$ occur simultaneously within the time interval of length D, from the one in which it doesn't. First we prove that no more than one simultaneous jump can occur within the interval $(0, D]$.

Corollary 1: Given the system (8) and (9) satisfying Assumption 1, for every $\bar{\tau} \in [0,T_1) \times [0,T_2)$, the set $\mathscr{T}_D(\bar{\tau})$ contains at most one element t_k such that $t_k = t_{k+1}$.

Proof: Suppose by contradiction that, there are more than one element in $\mathscr{T}_D(\bar{\tau})$ such that $t_k = t_{k+1}$ and consider two of them. Denote them t_j and t_i with $j < i$, and $\bar{n}_1$ and $\bar{n}_2$ the jumps of type T_1 and T_2 respectively, that occur in $(t_j, t_i]$. By hypothesis, $\bar{n}_1 \geq 1$ and $\bar{n}_2 \geq 1$. Since $t_j - t_i$ must be smaller than D, then $\bar{n}_1 < n_1$ and $\bar{n}_2 < n_2$. Thus it follows that $t_i - t_j = T_1\bar{n}_1 = T_2\bar{n}_2$ with $\bar{n}_1 < n_1$ and $\bar{n}_2 < n_2$, that contradicts Assumption 1. ■

Consider first the case in which the simultaneuos jump occurs at one jump instant, denote it t_j with $j \in \mathbb{N}_{n_D}$. The jump map takes the set value $\{(A_1x,\, 0,\, \tau_2) \cup (A_2x,\, \tau_1,\, 0)\}$ and the following jump instant is $t_{j+1} = t_j$. Thus the discrete-time dynamics of x is given by the following difference inclusion

$$x(k+1) \in A_D(\bar{\tau})x(k), \tag{15}$$

if $t(0) = \bar{\tau}$, where we denoted $x(kD)$ as $x(k)$, with slight abuse of notation, and $A_D(\bar{\tau}) = A_D^{(12)}(\bar{\tau}) \cup A_D^{(21)}(\bar{\tau})$ with

$$\begin{aligned} A_D^{(12)}(\bar{\tau}) &\triangleq e^{A_c\delta_{n_D+1}} \prod_{k\in j+2}^{n_D} A_{q_k} e^{A_c\delta_k} A_2 A_1 e^{A_c\delta_j} \prod_{k\in 1}^{j-1} A_{q_k} e^{A_c\delta_k} \\ A_D^{(21)}(\bar{\tau}) &\triangleq e^{A_c\delta_{n_D+1}} \prod_{k\in 1}^{n_D} A_{q_k} e^{A_c\delta_k} A_1 A_2 e^{A_c\delta_j} \prod_{k\in 1}^{j-1} A_{q_k} e^{A_c\delta_k} \end{aligned} \tag{16}$$

where $\delta_k = t_k - t_{k-1}$, with $\tau_0(\bar{\tau}) = 0$ and $\delta_{n_D+1}(\bar{\tau}) = \min\{\bar{\tau}_1,\, \bar{\tau}_2\}$ and $q_k \in \{1,2\}$. In the case when simultaneous jumps do not occur before D the jump map (9) takes values that are singletons. Then the dynamics of the state x at the time instants kD with $k \in \mathbb{N}$ is (15) with

$$A_D(\bar{\tau}) = A_D^{(0)}(\bar{\tau}) \triangleq e^{A_c\delta_{n_D+1}} \prod_{k\in 1}^{n_D} A_{q_k} e^{A_c\delta_k}. \tag{17}$$

Indeed, the dynamics reduces in this case to a difference equation rather than an inclusion. Then, summarizing, the discrete-time dynamics are given by (15) with

$$A_D = \begin{cases} A_D^{(12)} \cup A_D^{(21)} & \text{if } \exists j \in \mathbb{N}_{n_D-1} : t_j = t_{j+1}, \\ A_D^{(0)}, & \text{otherwise,} \end{cases} \tag{18}$$

where the dependence on $\bar{\tau}$ has been avoided for notational simplicty.

Of course, A must be Schur for all $A \in A_D(\bar{\tau})$ for the discrete-time system (15) to stay bounded and converge to the origin. Notice that such condition is only necessary, the necessary and sufficient one will be discussed in the sequel. The fact that $A_D(\bar{\tau})$ contains Schur or not Schur matrices might depend on the initial value $\bar{\tau}$, as illustrated in the example below.

Example 2: Consider the system (7) with matrices

$$A_c = \begin{bmatrix} 0 & -2\pi \\ 2\pi & 0 \end{bmatrix},\ A_1 = \begin{bmatrix} a-\varepsilon & 0 \\ 0 & a^{-1} \end{bmatrix},\ A_2 = \begin{bmatrix} a^{-1} & 0 \\ 0 & a-\varepsilon \end{bmatrix}$$

with $a > 1$, $\varepsilon > 0$ such that $a - \varepsilon > 1$ and $T_1 = T_2 = 0.5$. Recall that

$$e^{A_c t} = \begin{bmatrix} \cos(2\pi t) & -\sin(2\pi t) \\ \sin(2\pi t) & \cos(2\pi t) \end{bmatrix}.$$

For initial condition $\bar{\tau} = (0.25,\ 0)$ one has

$$A_D(\bar{\tau}) = A_2 e^{A_c 0.25} A_1 e^{A_c 0.25} = \begin{bmatrix} -a^{-2} & 0 \\ 0 & -(a-\varepsilon)^2 \end{bmatrix},$$

which is not Schur from $a - \varepsilon > 1$. On the other hand, setting $\bar{\tau} = (0,\ 0)$ and noticing that $A_1A_2 = A_2A_1$, one gets

$$A_D(\bar{\tau}) = A_2 A_1 e^{A_c 0.5} = \begin{bmatrix} -a^{-1}(a-\varepsilon) & 0 \\ 0 & -a^{-1}(a-\varepsilon) \end{bmatrix},$$

whose eigenvalues are $a^{-1}(a-\varepsilon) < a^{-1}a = 1$ and then the matrix is Schur.

An interesting issue is whether all $A \in A_D(\bar{\tau})$ are Schur for every possible initial condition $\bar{\tau} \in [0,T_1) \times [0,T_2)$. The property below is used in the following.

Proposition 2: Given the system (8) and (9) satisfying Assumptions 1 and 2, the spectrum of the matrices $A_D^{(0)}(\bar{\tau} + d\mathbb{1})$, $A_D^{(12)}(\bar{\tau} + d\mathbb{1})$ and $A_D^{(21)}(\bar{\tau} + d\mathbb{1})$ does not depend on $d \in \mathbb{R}$.

Proof: We consider in the proof the matrix $A_D^{(0)}(\bar{\tau})$, analogous reasonings hold for the matrices $A_D^{(12)}(\bar{\tau})$ and $A_D^{(21)}(\bar{\tau})$ as in (16). Suppose first that $d \in [0,D]$. By construction we have that $A_D^{(0)}(\bar{\tau}) = A_{D-d}^{(0)}(\bar{\tau} + d\mathbb{1})A_d(\bar{\tau})$. Moreover $A_D^{(0)}(\bar{\tau} + d\mathbb{1}) = A_d^{(0)}(\bar{\tau} + D\mathbb{1})A_{D-d}^{(0)}(\bar{\tau} + d\mathbb{1})$ and, since $A_d^{(0)}(\bar{\tau} + D\mathbb{1}) = A_d^{(0)}(\bar{\tau})$, then

$$\begin{aligned} \sigma\big(A_D^{(0)}(\bar{\tau})\big) &= \sigma\big(A_{D-d}^{(0)}(\bar{\tau} + d\mathbb{1})A_d^{(0)}(\bar{\tau})\big) \\ &= \sigma\big(A_d^{(0)}(\bar{\tau})A_{D-d}^{(0)}(\bar{\tau} + d\mathbb{1})\big) = \sigma\big(A_D^{(0)}(\bar{\tau} + d\mathbb{1})\big), \end{aligned}$$

where the second equality follows from the fact that $\sigma(AB) = \sigma(BA)$ for every $A, B \in \mathbb{R}^{n\times n}$. Finally, if $d \notin [0,\ D]$, there exists $k \in \mathbb{Z}$ such that $kD \leq d \leq (k+1)D$ which means that $\bar{d} = (d - kD) \in [0,\ D]$ and then, applying iteratively the result above, one has

$$\begin{aligned} \sigma\big(A_D^{(0)}(\bar{\tau})\big) &= \sigma\big(A_D^{(0)}(\bar{\tau} + kD\mathbb{1})\big) \\ &= \sigma\big(A_D^{(0)}(\bar{\tau} + kD\mathbb{1} + \bar{d})\big) = \sigma\big(A_D^{(0)}(\bar{\tau} + d\mathbb{1})\big). \end{aligned}$$

■

Geometrically, it means that analyzing the stability property of $A_D(\bar{\tau})$ implies analyzing the same property for every $(\tau_1,\, \tau_2)$ along the line starting from $\bar{\tau}$.

From Proposition 2, the analysis can be limited to the points, for instance, such that either $\tau_1 = 0$ and $\tau_2 \geq 0$ or $\tau_2 = 0$ and $\tau_1 \geq 0$.

Proposition 3: Given the system (8) and (9) satisfying Assumption 1, the matrices A with $A \in A_D(\bar{\tau})$ are Schur for every $\bar{\tau} \in [0,T_1) \times [0,T_2)$, if and only if they are Schur for every $\bar{\tau} \in [0,\, T_1) \times 0$ or for every $\bar{\tau} \in 0 \times [0,\, T_2)$.

Proof: Necessity is clear. For sufficiency, suppose that A is Schur for every $A \in A_D(\bar{\tau})$ for all $\bar{\tau} \in [0,T_1) \times 0$, the case $\bar{\tau} \in 0 \times [0,T_2)$ is analogous. For every $\bar{\tau} \in [0,T_1) \times [0,T_2)$ there exists $d \in (0,T_2]$ such the jump of type 2 occurs at time d. Then $\tau_2 + d = kT_2$ and $\bar{\tau} + \mathbb{1}d \in [0,\, T_1) \times 0$ with $k \in \mathbb{N}$. Thus, from Proposition 2, since the matrices of $A_D(\bar{\tau})$ and $A_D(\bar{\tau} + \mathbb{1}d)$ have the same spectrum and the latter are Schur by assumption then every $A \in A_D(\bar{\tau})$ is Schur. ■

C. Further reduction of the time-domain

Hereafter it will be proved that the stability analysis can be further reduced to some subset of $[0, T_1) \times 0$.

Lemma 1: Given T_1 and T_2 such that Assumption 1 holds, for every initial condition $\tau(0) = \bar{\tau}$ of (8) with $\bar{\tau} \in [0,\ T_1) \times 0$ and every $a \in \mathbb{N}$ it follows that

$$\begin{cases} \tau(aT_2) = \bar{\tau} & \text{if } a = mn_2, \\ |\tau(aT_2) - \bar{\tau}| \geq \dfrac{T_1}{n_2} & \text{otherwise.} \end{cases} \tag{19}$$

with $m \in \mathbb{N}$.

Proof: First notice that $\tau_2(aT_2) = \tau_2(0) = \bar{\tau}_2$ for all $a \in \mathbb{N}$ then it is sufficent to consider $|\tau_1(aT_2) - \bar{\tau}_1|$, in particular the condition on a such that

$$|\tau_1(aT_2) - \bar{\tau}_1| < \frac{T_1}{n_2},$$

or equivalently

$$\bar{\tau}_1 - \frac{T_1}{n_2} < \tau_1(aT_2) < \bar{\tau}_1 + \frac{T_1}{n_2}. \tag{20}$$

Since $\tau_1(aT_2) = (\bar{\tau}_1 + aT_2) \bmod (T_1)$ and $T_2 = T_1 n_1/n_2$, from Assumption 1, then (20) holds only if there exists $b \in \mathbb{N}$ such that

$$\begin{aligned} \bar{\tau}_1 - \frac{T_1}{n_2} + bT_1 < \bar{\tau}_1 + a\frac{n_1}{n_2}T_1 < \bar{\tau}_1 + \frac{T_1}{n_2} + bT_1 \\ \Leftrightarrow \quad -1 < an_1 - n_2 b < +1. \end{aligned} \tag{21}$$

Being $an_1 - n_2 b$ an integer, then condition (21) is satisfied if and only if $an_1 - n_2 b = 0$, that is if $a = n_2(b/n_2)$. Then (20) cannot hold if there is not $m = (b/n_2) \in \mathbb{N}$ such that $a = mn_2$, which proves the second part in (19). Moreover, if $a = mn_2$ with $m \in \mathbb{N}$ then $\tau_1(aT_2) = \tau_1(mn_2T_2) = \tau_1(mn_1T_1)$ and then $\tau_1(aT_2) = \bar{\tau}_1$, which terminates the proof. ■

Geometrically, the Lemma 1 means that, if the trajectory of τ is considered in the extended space, as in Figure 2, right, then at the time of the first $n_2 - 1$ jump of type 2, τ_1 is not in the open neighborhood of witdh T_1/n_2 around $\bar{\tau}_1$. Moreover, at the n_2-th jump of type 2, one has $t(n_2T_2) = \bar{\tau}$.

Remark 3: From Lemma 1 it follows that the only initial condition $\bar{\tau} \in [0, T_1/n_2) \times 0$ for which simultaneous jumps occur is $\bar{\tau} = (0, 0)$.

Proposition 4: Given T_1 and T_2 such that Assumption 1 holds, for every initial condition $\tau(0) = \bar{\tau}$ of (8) with $\bar{\tau} \in T_{1,2}(k) = [(k-1)T_1/n_2,\ kT_1/n_2) \times 0$, for $k \in \mathbb{N}_{n_2}$, one has that $\tau(aT_2) \notin T_{1,2}$ for all $a \in \mathbb{N}_{n_2-1}$ and $\tau(n_2T_2) = \bar{\tau}$. Moreover, the jump sequence associated with any $\tau(0) \in T_{1,2}(k)$ is the same.

The matrix $A_D(\bar{\tau})$, as in Definition 1, can be written in the following form

$$A_D(\bar{\tau}) = \begin{cases} A_D^{(12)}(\bar{\tau}) \cup A_D^{(21)}(\bar{\tau}) & \text{if } \bar{\tau} = (0,0), \\ \prod\limits_{k \in \mathbb{N}_m} \mathscr{R}_k e^{\mathscr{A}_k \delta}, & \text{otherwise,} \end{cases} \tag{22}$$

for all $\bar{\tau} = (\delta,\ 0) \in [0,\ T_1/n_2) \times 0$, with $m \leq n_D$, for appropriate matrices $\mathscr{R}_k, \mathscr{A}_k \in \mathbb{R}^{n \times n}$ for $k \in \mathbb{N}_m$. The following example shows how to build the matrix representation of $A_D(\bar{\tau})$ as in (22).

Example 3: Consider the system (8) and (9) satisfying Assumption 1, with $T_1 = 3$ and $T_2 = 7$ and $\bar{\tau} = (\delta,\ 0) \in [0,\ 1) \times 0$. Then, $n_1 = 7, n_2 = 3$ and $D = 21$. The graphical representation of the trajectory of the states τ_1 and τ_2 is depicted in Figure 2, right. From geometrical inspection we have that, for $\delta = 0$, one has

$$\begin{aligned} A_D^{(12)}(\bar{\tau}) &= A_2 A_1 e^{A_c 3} A_1 e^{A_c 3} A_1 e^{A_c} A_2 e^{A_c 2} \\ &\quad \cdot A_1 e^{A_c 3} A_1 e^{A_c 2} A_2 e^{A_c} A_1 e^{A_c 3} A_1 e^{A_c 3}, \\ A_D^{(21)}(\bar{\tau}) &= A_1 A_2 e^{A_c 3} A_1 e^{A_c 3} A_1 e^{A_c} A_2 e^{A_c 2} \\ &\quad \cdot A_1 e^{A_c 3} A_1 e^{A_c 2} A_2 e^{A_c} A_1 e^{A_c 3} A_1 e^{A_c 3}, \end{aligned} \tag{23}$$

while

$$\begin{aligned} A_D^{(0)}(\bar{\tau}) &= A_2 e^{A_c \delta} A_1 e^{A_c 3} A_1 e^{A_c 3} A_1 e^{A_c} e^{-A_c \delta} A_2 e^{A_c 2} e^{A_c \delta} \\ &\quad \cdot A_1 e^{A_c 3} A_1 e^{A_c 2} e^{-A_c \delta} A_2 e^{A_c} e^{A_c \delta} A_1 e^{A_c 3} A_1 e^{A_c 3} e^{-A_c \delta}, \end{aligned}$$

for $\delta \in (0,\ 1)$, that can be rewritten as in (22) with $m = 6$ and

$$\begin{aligned} &\mathscr{A}_1 = -A_c,\ \mathscr{A}_2 = A_c,\ \mathscr{A}_3 = -A_c, \\ &\mathscr{A}_4 = A_c,\ \mathscr{A}_5 = -A_c,\ \mathscr{A}_6 = A_c, \\ &\mathscr{R}_1 = A_1 e^{A_c 3} A_1 e^{A_c 3},\ \mathscr{R}_2 = A_2 e^{A_c},\ \mathscr{R}_3 = A_1 e^{A_c 3} A_1 e^{A_c 2}, \\ &\mathscr{R}_4 = A_2 e^{A_c 2},\ \mathscr{R}_5 = A_1 e^{A_c 3} A_1 e^{A_c 3} A_1 e^{A_c},\ \mathscr{R}_6 = A_2. \end{aligned}$$

IV. Stability analysis: necessary and sufficient conditions

Before providing the main result of the paper, concerning the constructive necessary and sufficient conditions for global exponential stability, a property is given that is functional for the objective.

Definition 2: The convex compact set $\Omega \subseteq \mathbb{R}^n$ with $0 \in \operatorname{int} \Omega$ is a λ-contractive set, with $\lambda \in [0, 1)$, for the difference inclusion $x^+ \in \bar{A}x$ with $\bar{A} \subseteq \mathbb{R}^{n \times n}$ if $A\Omega \subseteq \lambda\Omega$ for all $A \in \bar{A}$.

The following result comes from the literature, [14], [3].

Lemma 2: The difference inclusion $x^+ \in \bar{A}x$ is globally exponentially stable if and only if there exists a polytopic λ-contractive set.

The set of symmetric positive definite matrices in $\mathbb{R}^{n \times n}$ is denoted $\mathbb{S}^n$.

Theorem 1: The compact set $\mathscr{A}$ defined in (10) is globally exponentially stable for the system (8) with (9) satisfying Assumption 1 with decay rate $\lambda \in [0, 1)$, if and only if there exits $P_i : (0, T_1/n_2) \to \mathbb{S}^n$ for $i \in \mathbb{N}_{m+1}$ such that

$$e^{\mathscr{A}_i^T \delta} \mathscr{R}_i^T P_{i+1}(\delta) \mathscr{R}_i e^{\mathscr{A}_i \delta} \leq P_i(\delta), \qquad \forall i \in \mathbb{N}_m, \tag{24}$$

and $P_1(\delta) \leq \lambda^2 P_{m+1}(\delta)$, with $\bar{\tau} = (\delta,\ 0)$, for all $\delta \in (0, T_1/n_2)$, with $\mathscr{R}_k, \mathscr{A}_k$, for $k \in \mathbb{N}_m$ as in (22) and a λ-contractive polytope for $\bar{A} = A_D^{(12)}((0,0)) \cup A_D^{(21)}((0,0))$.

Proof: Due to space limitations we do not provide the proof here. ■

The condition given in Theorem 1 is necessary and sufficent for stability of system (8) and (9), for every initialization of $(\tau_1, \tau_2) \in [0, T_1) \times [0, T_2)$, but it concerns an infinite number of LMI conditions. Since our aim is to provide a constructive method, a computation-oriented equivalent result is given hereafter.

Proposition 5: The compact set $\mathscr{A}$ defined in (10) is globally exponentially stable for the system (8) with (9) satisfying Assumption 1 with decay rate $\lambda \in [0,1)$, if and only if for all $\nu \in (\lambda, 1)$ there exist: $\mu \in \mathbb{R}$; $J \in \mathbb{N}$; $P_{i,j} \in \mathbb{S}^n$ and $\alpha_{i,j} \in \mathbb{R}$ for $i \in \mathbb{N}_{m+1}$ and $j \in \mathbb{N}_J$ such that

$$\begin{array}{ll} e^{\mathscr{A}_i^T \delta_j} \mathscr{R}_i^T P_{i+1,j} \mathscr{R}_i e^{\mathscr{A}_i \delta_j} \leq \mu^2 P_{i,j}, & \forall i \in \mathbb{N}_m,\ j \in \mathbb{N}_J, \\ \mathscr{A}_i^T P_{i,j} + P_{i,j} \mathscr{A}_i \leq \alpha_{i,j} P_{i,j}, & \forall i \in \mathbb{N}_m,\ j \in \mathbb{N}_J, \\ \max\{1,\ e^{\sqrt{\alpha_{i,j}}\Delta}\} \leq \mu^{-1}, & \forall i \in \mathbb{N}_m,\ j \in \mathbb{N}_J, \\ \delta_{j+1} = \delta_j + \Delta, & \forall j \in \mathbb{N}_J, \end{array} \tag{25}$$

with $\Delta = T_1/(n_2 J)$; and $P_{1,j} \leq \nu^2 P_{m+1,j}$, with $\mathscr{R}_k, \mathscr{A}_k$, for $k \in \mathbb{N}_m$ as in (22) and a λ-contractive polytope for $\bar{A} = A_D^{(12)}((0,0)) \cup A_D^{(21)}((0,0))$.

Proof: The proof consists in demonstrating that (24) and (25) are equivalent. An analogous result has been proved in [6]. ■

V. Numerical example

Although a direct comparison is not possible since the problems are not the same, we inspire our example from the literature, in particular from [4], [13]. The plant dynamics is given by

$$A_p = \begin{bmatrix} 0.1 & 1 \\ 0 & -0.1 \end{bmatrix},\ B_p = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix},\ C_p = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

which is analogous to an example treated in [4], [13] but we replaced the pole in 0 with one in 0.1 and the state is supposed to be not accessible. The dynamical output-feedback controller is composed by an observer whose state feeds the gain $K = [-3.75\ \ -11.5]$ employed in the cited papers. Then the controller results given by

$$A_k = A_p + B_p K + L C_p, \quad B_k = -L, \quad C_k = K,$$

where L is the observer gain designed such that $A_p + L C_p$ is Hurwitz. The overall system is exponentially stable in closed loop. The matrices A_c, A_1 and A_2 in (7) are obtained by using the numerical values given above in equations (4)-(6).

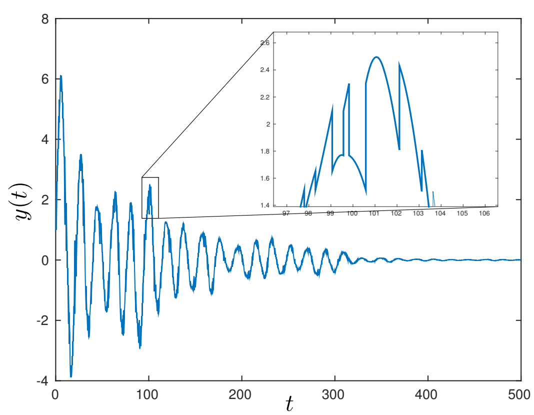

Fig. 3. Time evolution of the system ouput with $T_1 = 3 \cdot 0.254 = 0.7620$ and $T_2 = 7 \cdot 0.254 = 1.7780$

We consider the asynchronous sampling times $T_1 = 3a$ and $T_2 = 7a$ and try to maximize a such that the resulting sampled-data system is exponentially stable for every possible initialization of the clocks. We found that one among the matrices $A_D(\bar{\tau})$ has a spectral radius of 0.9989 for $a = 0.254$ and of 1.0076 for $a = 0.255$. Applying our method we found that condition (25) with $a = 0.254$ is satisfied with $J = 4000$ and the maximal contraction obtained is 0.9999. Moreover, since A_1 and A_2 commute, than $A_D^{(12)} = A_D^{(21)}$ and then testing that one of them is Schur, which is the case, is sufficient. Finally we run a simulation with $a = 0.254$ to test the convergence of the system, see Figure 3.

VI. Conclusions

In this paper we presented a constructive method to check whether a linear closed-loop system with asynchronous input and output sampling is globally exponentially stable. The approach is based on a computation-oriented necessary and sufficient condition leading to a finite set of convex constraints. The result presented is also a preliminary step for the analysis of systems with uncertain sampling periods.

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